



The torsional buckling and writhing of a simply supported rod hanging under gravity

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Abstract

The problem of a finite-length simply supported rod hanging under gravity and subject to a prescribed tangential twist Tw is studied using asymptotic and numerical methods. A three-dimensional formulation of the problem is given in which a small parameter ε^2 measures the relative sizes of bending and gravitational forces. For small values of Tw , the rod shape is found by singular perturbation methods and consists of an *outer* catenary-like solution and an *inner* boundary layer solution. Large twist $Tw = O(1/\varepsilon)$ of an almost straight rod produces a torque on the order of the Greenhill buckling level and is shown numerically to cause buckling into a modulated helix-like spiral with period of $O(\varepsilon)$ superimposed onto a parabolic sag across the spanned distance. Multiple scale methods are used in this parameter regime to obtain an approximate description of the postbuckled solution. This analysis is found to capture all the broad features indicated by the numerics. As Tw is further increased, the deformation may localise and the rod jump into a self-intersecting writhed shape. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The study of the buckling of long, thin, straight rods under applied axial tension and moment goes at least as far back as Kirchhoff (1859). Such problems arise in many applications in engineering, and physical sciences, ranging from the failure of undersea cables and pipelines, e.g. Coyne (1990), to the helical spiral inversion seen in the tendrils of climbing plants (Goriely and Tabor, 1998). Greenhill (1883) showed that under only remote tension T and moment Q , a rod of length L first buckles into a helical shape when

$$\frac{4\pi^2 B^2}{L^2} = Q^2 - 4BT, \quad (1)$$

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where B is the bending rigidity of the rod. The helical solution was further analysed in the treatise by Love (1927). Also, see Timoshenko and Gere (1961) for a more modern engineering treatment, and the book by Antman (1995) for a mathematical account of the general theory of rod mechanics.

It is well known that in the absence of torsion, a rod hanging under gravity while being pulled horizontally will hang in a catenary. If torsion is applied, phenomenological experiments, e.g. Thompson and Champneys (1996), indicate that if the tension is sufficiently high, the effect of gravity appears not to destroy the initial buckling into a helix. Indeed, in Champneys et al. (1997), gravity was written as one of several perturbations to the ‘perfect’ problem, but, unlike nonsymmetric cross-section, intrinsic curvature and shear deformation, the effect of gravity was not analysed. The form of the perturbation caused by gravity was much harder, because with the others, the force and moment equilibrium equations when written in a material coordinate basis could be analysed as an autonomous dynamical system, which is decoupled from the *Frenet–Seret* equations for the centreline of the rod. Because this is not possible in the case of gravity, one cannot use the trick of passing to an infinitely long rod and employing the (Kirchhoff) analogy with an initial-value problem (see Kehrbaum and Maddocks (1997) for the details of this analogy).

A recent treatment by Stump and van der Heijden (2000) provides the starting point for the present analysis. Motivated by the problem of an underwater pipeline being laid from a barge, they study a finite-length rod, clamped at both ends, subject to tension and a moderate amount of twist supplied by rotations of the barge. A nondimensionalisation introduces a small parameter ϵ , which is the square root of the ratio of characteristic bending and gravitational forces. As the total twist Tw inserted into the rod is $O(1)$, the solution is represented as a matched asymptotic expansion with an *outer* catenary-like layer and *inner* bending boundary layers on the $O(\epsilon)$ length scale.

The present paper considers a (symmetric, unshearable) rod subject to simply supported boundary conditions. The general problem is formulated for an arbitrary span length of d and arbitrary levels of inserted twist Tw . For the Tw of $O(1)$, the boundary-layer analysis of Stump and van der Heijden (2000) is adapted for the present boundary conditions, giving a catenary-like solution in the centre of the span with boundary layers at the ends. The truncated analytical results agree well with numerical computations. This solution breaks down at $\text{Tw} = O(1/\epsilon)$.

Accurate numerics are used to explore the effects of large Tw values. For small ϵ , the solution remains essentially a catenary until a value of Tw consistent with the Greenhill buckling torque is reached. At this point, the solution changes markedly and is given approximately by a catenary and a superimposed helix with a period of $O(\epsilon)$. As Tw is increased further, the helix angle tightens and the centre of the span rises. Eventually, after all the slack is taken up, the helical deformations begin to localise. Motivated by these numerical results, a heuristic multiple-scale analysis in the outer region is used to separate the two leading components of the solution. The results explain all the broad features of the postbuckling path.

The paper is ordered as follows: Section 2 develops the mathematical model for the rod. Section 3 considers the low-twist case, where $\text{Tw} = O(1)$. Results obtained by asymptotic methods are compared with numerical work. Section 4 deals with the high-twist case for small ϵ . Detailed numerical results are presented first, which suggest the scalings for the multiple-scale analysis, which then follows. Section 5 concludes the study with some remarks.

2. The model

A long, thin, inextensible, unshearable rod of natural length L and with circular cross-section is assumed to span the x axis over the region $(0 \leq x \leq d)$ of a Cartesian coordinate system, as shown in Fig. 1. The arclength coordinate s ($0 \leq s \leq L$) runs along the rod, and gravity points in the negative \mathbf{j} direction. The configuration of the rod’s centreline is specified by the position vector $\mathbf{R}(s)$, and the rod is assumed to have a total twist Tw . For simplicity, the rod is taken as linear elastic.

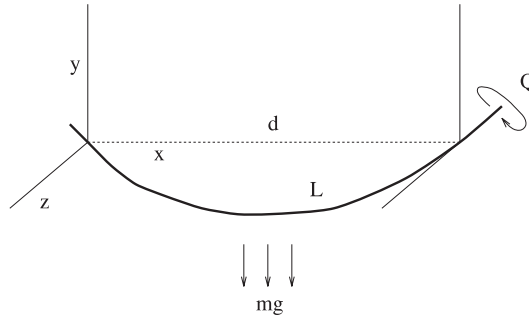


Fig. 1. Sketch of the physical set-up.

As the cross-section is circular, the equilibrium equations for such a rod can be formulated entirely in terms of the Frenet derivatives of its centreline without the need for the more general Cosserat formulation. Thus, the equations for force and moment balance governing the equilibrium shape of the rod with centreline $\mathbf{R}(s)$ are written as

$$(T\mathbf{R}')' + \mathbf{V}' - mg\mathbf{j} = 0, \quad (2a)$$

$$(Q\mathbf{R}')' + \mathbf{M}' + \mathbf{R}' \times \mathbf{V} = 0, \quad (2b)$$

where, $T(s)$ is the tensile component of force, $\mathbf{V}(s)$, the shear force, mg , the weight per unit length of rod, $\mathbf{M}(s)$, the bending moment; and Q , the twisting moment. These equations are augmented with the inextensibility condition, the normality of the shear force and the tangent vectors, and the linear constitutive law linking bending moment and curvature:

$$\mathbf{R}' \cdot \mathbf{R}' = 1, \quad (3a)$$

$$\mathbf{R}' \cdot \mathbf{V} = 0, \quad (3b)$$

$$\mathbf{M} = B(\mathbf{R}' \times \mathbf{R}''). \quad (3c)$$

It is straightforward to show that the torque Q is constant along the rod by forming the dot product of \mathbf{R}' with Eq. (2b) to obtain $Q' = 0$. The rate of cross-sectional rotation N' and the twist Tw are linked to Q by the constitutive law:

$$Q = KN' = \frac{2\pi K \text{Tw}}{L}, \quad (4)$$

where, $B = EI$ and $K = GJ$ are the bending and torsional rigidities, respectively, E is Young's modulus, G , the shear modulus, $I = \pi a^4/4$, the second moment of area (a being the radius of the rod), and $J = \pi a^4/2$, the polar moment of inertia.

Eqs. (2a)–(4) may be made dimensionless against the characteristic length L and force mgL by setting

$$\bar{s} = \frac{s}{L}, \quad \bar{\mathbf{M}} = \frac{\mathbf{M}}{mgL^2}, \quad \bar{Q} = \frac{Q}{mgL^2}, \quad \bar{T} = \frac{T}{mgL}, \quad \bar{\mathbf{V}} = \frac{\mathbf{V}}{mgL},$$

and by introducing the dimensionless parameters,

$$\varepsilon^2 = \frac{B}{mgL^3} \quad \text{and} \quad \kappa = \frac{K}{B}. \quad (5)$$

The value of $\kappa = 1/(1 + \nu)$ (ν is Poisson's ratio) is of $O(1)$, whereas the *stiffness* parameter ε is typically $\ll O(1)$.

Henceforth, we deal exclusively with dimensionless variables and therefore drop the overbars. The dimensionless equations then become

$$(TR')' + V' - j = 0, \quad (6a)$$

$$(QR')' + M' + R' \times V = 0, \quad (6b)$$

$$R' \cdot R' = 1, \quad (6c)$$

$$R' \cdot V = 0, \quad (6d)$$

$$M = \varepsilon^2(R' \times R''), \quad (6e)$$

$$Q = \varepsilon^2 \kappa N' = \varepsilon^2 (2\pi \kappa Tw). \quad (6f)$$

This system is reduced to a single vector equation by first integrating Eq. (6a) once to obtain

$$TR' + V - sj + C = 0, \quad (7)$$

where C is as yet an unknown vector. Next, formation of the scalar product of Eq. (7) with R' along with the use of Eq. (6d) and the inextensibility condition (6c) leads to the expression for the tension:

$$T = s(R' \cdot j) - C \cdot R'. \quad (8)$$

Finally, the use of Eqs. (7) and (8) to eliminate V and T from Eq. (6b) followed by the formation of the vector product with R' produces the single second-order equation for R' :

$$\varepsilon^2 [2\pi \kappa Tw(R' \times R'') - R''' - R'(R'' \cdot R'')] + [s(R' \cdot j) - C \cdot R']R' - sj + C = 0. \quad (9)$$

This study is specifically interested in the dimensionless simply supported boundary conditions,

$$R(0) = 0, \quad R'(0) = 0, \quad R(1) = d i, \quad R'(1) = 0 \quad (10)$$

in which the bending curvature vanishes at the ends of the rod. Eq. (9) describes the shape of the rod subject to the boundary conditions (10). Also, we restrict the consideration to shapes that are symmetric with respect to the centre of the span, as shown in Fig. 1.

3. Low-twist analysis

In this section, Eq. (9) is analysed under the assumption that Q is $O(\varepsilon^2)$, or Tw is $O(1)$. The analysis closely follows Stump and van der Heijden (2000) to which we refer for more details. As the equation obtained by substituting $\varepsilon = 0$ into Eq. (9) is of a lower order than Eq. (9) itself, the boundary conditions (10) cannot in general be satisfied, and we expect $O(\varepsilon)$ boundary layers near both ends of the rod. In the following, we first analyse the solution away from the ends (the *outer* solution), and then, after introducing the scaled distance $\xi = s/\varepsilon$, study the solution in the boundary layers (the *inner* solutions). Finally, the solutions are patched together by matching with an intermediate coordinate.

3.1. The outer solution

The quantities $R(s)$ and C are expanded in regular perturbation series in the parameter ε :

$$R(s) = R_0(s) + \varepsilon R_1(s) + \varepsilon^2 R_2(s) + \dots, \quad (11a)$$

$$C = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \dots, \quad (11b)$$

which are substituted into Eq. (9) to obtain a succession of equations in powers of ϵ , the first two of which are

$$O(1) : [s(\mathbf{R}'_0 \cdot \mathbf{j}) - \mathbf{C}_0 \cdot \mathbf{R}'_0] \mathbf{R}'_0 - s\mathbf{j} + \mathbf{C}_0 = 0, \quad (12a)$$

$$O(\epsilon) : s(\mathbf{R}'_0 \cdot \mathbf{j}) \mathbf{R}'_1 + s(\mathbf{R}'_1 \cdot \mathbf{j}) \mathbf{R}'_0 - (\mathbf{C}_1 \cdot \mathbf{R}'_0) \mathbf{R}'_0 - (\mathbf{C}_0 \cdot \mathbf{R}'_1) \mathbf{R}'_0 - (\mathbf{C}_0 \cdot \mathbf{R}'_0) \mathbf{R}'_1 + \mathbf{C}_1 = 0. \quad (12b)$$

The use of the inextensibility condition (3a) provides the additional constraints $\mathbf{R}'_0 \cdot \mathbf{R}'_0 = 1$ and $\mathbf{R}'_0 \cdot \mathbf{R}'_1 = 0$.

The $O(1)$ solution of Eq. (12a) and (12b) is obtained by introducing $\mathbf{R}_0 = x_0 \mathbf{i} + y_0 \mathbf{j} + z_0 \mathbf{k}$ and $\mathbf{C}_0 = c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k}$ into Eq. (12a) and solving the resulting set of equations by elementary means to obtain

$$\begin{aligned} x_0 &= -|c_x| \sinh^{-1} \left\{ \frac{c_y - s}{\sqrt{c_x^2 + c_z^2}} \right\} + |c_x| \sinh^{-1} \left\{ \frac{c_y}{\sqrt{c_x^2 + c_z^2}} \right\} + X_0, \\ y_0 &= -\alpha \sqrt{c_x^2 + (c_y - s)^2 + c_z^2} + \alpha \sqrt{c_x^2 + c_y^2 + c_z^2} + Y_0, \\ z_0 &= \frac{c_z}{c_x} x_0 + Z_0, \end{aligned} \quad (13)$$

where $\alpha = c_x/|c_x|$, and (X_0, Y_0, Z_0) are additional constants of integration. This is the three-dimensional catenary lying in a plane making an angle $\chi = \arctan(c_z/c_x)$ with the (\mathbf{i}, \mathbf{j}) coordinate plane. The constants (c_x, c_y, c_z) are still unknown and will be determined in the matching process. We can already observe, however, that the solutions come in pairs as in Eq. (13) they are invariant under a simultaneous sign change of c_x and c_z . (These correspond to the conventional downward hanging solution under tension, and the inverted arch-like solution under compression.)

The solution to the $O(\epsilon)$ equation is found from the formation of two successive vector products of \mathbf{R}'_0 with Eq. (12b), which yields

$$\mathbf{R}'_0 \times (\mathbf{R}'_0 \times \mathbf{R}'_1) [s(\mathbf{R}'_0 \cdot \mathbf{j}) - \mathbf{C}_0 \cdot \mathbf{R}'_0] + \mathbf{R}'_0 \times (\mathbf{R}'_0 \times \mathbf{C}_1) = 0. \quad (14)$$

The use of the triple vector product and inextensibility gives

$$\mathbf{R}'_1 = \frac{\mathbf{R}'_0(\mathbf{R}'_0 \cdot \mathbf{C}_1) - \mathbf{C}_1}{s(\mathbf{R}'_0 \cdot \mathbf{j}) - \mathbf{C}_0 \cdot \mathbf{R}'_0}. \quad (15)$$

The solution for $\mathbf{R}_1 = x_1 \mathbf{i} + y_1 \mathbf{j} + z_1 \mathbf{k}$ and $\mathbf{C}_1 = d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k}$ is obtained by using Eq. (13) along with elementary integrations to yield

$$\begin{aligned} \mathbf{R}_1(s) &= \{(-|c_x|c_z d_z + \alpha c_z^2 d_x)I_1(s) - |c_x|d_y I_2(s) + \alpha d_x I_3(s)\} \mathbf{i} + \{\alpha(c_x^2 d_y + c_z^2 d_y)I_1(s) \\ &\quad - (\alpha c_z d_z + |c_x|d_x)I_2(s)\} \mathbf{j} + \{(-|c_x|c_z d_x + \alpha c_x^2 d_z)I_1(s) - \alpha c_z d_y I_2(s) + \alpha d_z I_3(s)\} \mathbf{k} \\ &\quad + X_1 \mathbf{i} + Y_1 \mathbf{j} + Z_1 \mathbf{k}, \end{aligned} \quad (16)$$

where (X_1, Y_1, Z_1) are integration constants and

$$\begin{aligned} I_1(s) &= -\frac{c_y - s}{(c_x^2 + c_z^2)\sqrt{c_x^2 + (c_y - s)^2 + c_z^2}}, \\ I_2(s) &= \frac{1}{\sqrt{c_x^2 + (c_y - s)^2 + c_z^2}}, \\ I_3(s) &= \frac{c_y - s}{\sqrt{c_x^2 + (c_y - s)^2 + c_z^2}} - \sinh^{-1} \left(\frac{c_y - s}{\sqrt{c_x^2 + c_z^2}} \right). \end{aligned}$$

3.2. The inner solution

As we are interested in symmetric rod shapes, only the boundary layer near $s = 0$ needs to be considered. We introduce the scaled coordinate $\xi = s/\epsilon$ and write

$$\mathbf{R}(s) = \epsilon \hat{\mathbf{R}}(\xi) = \epsilon \hat{\mathbf{R}}_0(\xi) + \epsilon^2 \hat{\mathbf{R}}_1(\xi) + \dots, \quad (17)$$

so that the tangent has the form

$$\mathbf{R}'(s) = \hat{\mathbf{R}}'(\xi) = \hat{\mathbf{R}}'_0(\xi) + \epsilon \hat{\mathbf{R}}'_1(\xi) + \dots,$$

where the prime denotes differentiation with respect to the appropriate argument.

In the boundary layer, Eq. (9) is written in terms of $\hat{\mathbf{R}}(\xi)$ by

$$\left[\epsilon \xi (\hat{\mathbf{R}}' \cdot \mathbf{j}) - \mathbf{C} \cdot \hat{\mathbf{R}}' \right] \hat{\mathbf{R}}' - \hat{\mathbf{R}}''' - \hat{\mathbf{R}}'(\hat{\mathbf{R}}'' \cdot \hat{\mathbf{R}}'') + \epsilon 2\pi\kappa \text{Tw}(\hat{\mathbf{R}}' \times \hat{\mathbf{R}}'') - \epsilon \xi \mathbf{j} + \mathbf{C} = 0. \quad (18)$$

With Eq. (11b), Eq. (18) can be separated into a sequence of equations in ϵ , the first two of which are

$$\begin{aligned} \text{O}(1) : & (\mathbf{C}_0 \cdot \hat{\mathbf{R}}'_0) \hat{\mathbf{R}}'_0 + \hat{\mathbf{R}}_0''' + \hat{\mathbf{R}}'_0(\hat{\mathbf{R}}_0'' \cdot \hat{\mathbf{R}}_0'') = \mathbf{C}_0, \\ \text{O}(\epsilon) : & \xi (\hat{\mathbf{R}}'_0 \cdot \mathbf{j}) \hat{\mathbf{R}}'_0 - (\mathbf{C}_1 \cdot \hat{\mathbf{R}}'_0) \hat{\mathbf{R}}'_0 - (\mathbf{C}_0 \cdot \hat{\mathbf{R}}'_1) \hat{\mathbf{R}}'_0 \\ & - (\mathbf{C}_0 \cdot \hat{\mathbf{R}}'_0) \hat{\mathbf{R}}'_1 - \hat{\mathbf{R}}_1''' - \hat{\mathbf{R}}'_1(\hat{\mathbf{R}}_0'' \cdot \hat{\mathbf{R}}_0'') - 2\hat{\mathbf{R}}'_0(\hat{\mathbf{R}}_1'' \cdot \hat{\mathbf{R}}_0'') \\ & - \xi \mathbf{j} + 2\pi\kappa \text{Tw}(\hat{\mathbf{R}}'_0 \times \hat{\mathbf{R}}_0'') + \mathbf{C}_1 = 0. \end{aligned} \quad (19)$$

The inextensibility condition is also expanded and leads to a series of equations: $\hat{\mathbf{R}}'_0 \cdot \hat{\mathbf{R}}'_0 = 1$, $\hat{\mathbf{R}}'_0 \cdot \hat{\mathbf{R}}'_1 = 0$, and so on. It is noted that for $\text{O}(1)$ values of Tw , twist appears first in the $\text{O}(\epsilon)$ equation. The boundary conditions (10) become

$$\hat{\mathbf{R}}_0(0) = \hat{\mathbf{R}}_1(0) = \dots = 0, \quad \hat{\mathbf{R}}_0'(0) = \hat{\mathbf{R}}_1'(0) = \dots = 0. \quad (20)$$

The solution to the $\text{O}(1)$ equation is obtained by forming the vector product with $\hat{\mathbf{R}}'_0$ and integrating the result to get

$$\hat{\mathbf{R}}'_0 \times \hat{\mathbf{R}}_0'' = \hat{\mathbf{R}}_0 \times \mathbf{C}_0 + \mathbf{D}_0, \quad (21)$$

where \mathbf{D}_0 is a constant of integration. From Eq. (20) we have $\mathbf{D}_0 = 0$. Another cross product with $\hat{\mathbf{R}}'_0$ yields

$$\hat{\mathbf{R}}_0'' = -\hat{\mathbf{R}}'_0 \times [\hat{\mathbf{R}}_0 \times \mathbf{C}_0]. \quad (22)$$

This is the equation of the planar elastica. As no bending moments are applied at the rod ends and twist is absent at the current level of approximation, the leading order rod shape $\hat{\mathbf{R}}_0$ is confined to a plane containing \mathbf{i} , the direction of the span vector connecting the ends of the rod, and \mathbf{j} , the line of gravity. It is convenient to define a set of basis vectors $(\mathbf{N}, \mathbf{k}, \mathbf{P})$ by

$$\mathbf{N} = \alpha \frac{\mathbf{C}_0}{|\mathbf{C}_0|}, \quad \mathbf{P} = \mathbf{k} \times \mathbf{N} \quad (23)$$

so that for the inner solution, we can write

$$\hat{\mathbf{R}}_0(\xi) = \rho(\xi) \mathbf{P} + h(\xi) \mathbf{N}. \quad (24)$$

Inextensibility, the boundary conditions (20), and the requirement that \mathbf{R}'' vanish as $\xi \rightarrow \infty$ provide the limiting values

$$\rho(0) = h(0) = 0 \quad \text{and} \quad \lim_{\xi \rightarrow \infty} \rho(\xi) \rho'(\xi) = 0, \quad \lim_{\xi \rightarrow \infty} h'(\xi) = 1.$$

The introduction of Eq. (24) into Eq. (22) gives the component equations:

$$\frac{d^2\rho}{d\xi^2} = -\alpha|C_0|\rho\frac{dh}{d\xi}, \quad \frac{d^2h}{d\xi^2} = \alpha|C_0|\rho\frac{d\rho}{d\xi}. \quad (25)$$

Straightforward integration and use of the boundary conditions gives $\rho = 0$ and $h = \xi$, and hence,

$$\hat{\mathbf{R}}_0(\xi) = \xi\mathbf{N}. \quad (26)$$

We now proceed by solving the $O(\epsilon)$ equation in Eq. (19). The use of Eq. (26) in Eq. (19) and the formation of two successive vector products of the equation with \mathbf{N} gives

$$\hat{\mathbf{R}}_1''' + \alpha|C_0|\hat{\mathbf{R}}_1' = \mathbf{C}_1 - \xi\mathbf{j} - [\mathbf{C}_1 \cdot \mathbf{N} - \xi\mathbf{j} \cdot \mathbf{N}]\mathbf{N}. \quad (27)$$

We can again write

$$\hat{\mathbf{R}}_1(\xi) = \rho_1(\xi)\mathbf{P} + h_1(\xi)\mathbf{N}, \quad (28)$$

and substitution of this into Eq. (27) gives

$$\rho_1''' + \alpha|C_0|\rho_1' = \mathbf{P} \cdot [\mathbf{C}_1 - \xi\mathbf{j}], \quad (29a)$$

$$h_1''' + \alpha|C_0|h_1' = 0. \quad (29b)$$

Integration of the ρ_1 equation along with the restriction that the shape $\hat{\mathbf{R}}(\xi)$ remains a bounded function of s within the boundary layer (which means that ρ_1 can only grow as fast as ξ^2) and use of $\rho_1(0) = 0$ yields

$$\rho_1 = +\frac{(\mathbf{P} \cdot \mathbf{j})\xi^2}{2|C_0|} - \frac{(\mathbf{P} \cdot \mathbf{C}_1)\xi}{|C_0|} + \frac{\mathbf{P} \cdot \mathbf{j}}{|C_0|^2}(1 - e^{-\xi\sqrt{|C_0|}}). \quad (30)$$

After the integration of Eq. (29b) and the application of the boundary conditions $h_1(0) = 0$ and $h'(\infty) = 0$, the h_1 equation is found to have only the trivial solution, $h_1 = 0$, so that the complete inner solution up to $O(\epsilon^2)$ has the form:

$$\epsilon\hat{\mathbf{R}}(\xi) = \epsilon\xi\mathbf{N} + \epsilon^2\left[\frac{\xi^2}{2}\frac{(\mathbf{P} \cdot \mathbf{j})}{|C_0|} + \xi\frac{(\mathbf{P} \cdot \mathbf{C}_1)}{|C_0|} + \frac{(\mathbf{P} \cdot \mathbf{j})}{|C_0|^2}(1 - e^{-\xi\sqrt{|C_0|}})\right]\mathbf{P} + \dots \quad (31)$$

As only solutions which are symmetric with respect to $s = 1/2$ are under consideration, the boundary layer solution near $s = 1$ is obtained by reflecting the inner solution components about the mid-point of the span.

3.3. Matching the inner and outer solutions

The matching between the inner and outer solutions is conducted with the method of intermediate coordinates (Kevorkian and Cole, 1996). The arclength s and the boundary layer coordinate ξ are expressed in terms of an intermediate co-ordinate $\eta(\epsilon)$ according to

$$s = \eta\tau, \quad \xi = \frac{\eta\tau}{\epsilon},$$

where $\eta(\epsilon)$ is such that $\eta(\epsilon)$ and $\epsilon/\eta(\epsilon)$ are both $O(1)$, and τ is fixed. Then, the inner solution at $s = 0$ has the form:

$$\mathbf{R}(s) = \epsilon\hat{\mathbf{R}}_0\left(\frac{\eta\tau}{\epsilon}\right) + \epsilon^2\hat{\mathbf{R}}_1\left(\frac{\eta\tau}{\epsilon}\right) + \dots, \quad (32)$$

whereas the outer expansion becomes

$$\mathbf{R}(s) = \mathbf{R}_0(\eta\tau) + \epsilon\mathbf{R}_1(\eta\tau) + \dots \quad (33)$$

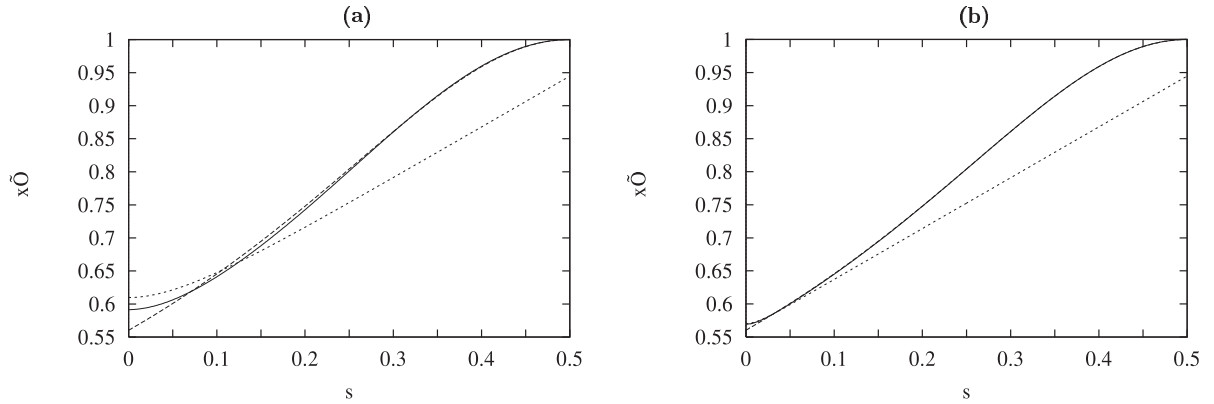


Fig. 2. Derivatives of the x components of the outer (---), inner (···) and exact (—) solution against arclength s for (a) $\epsilon = 0.05$ and (b) $\epsilon = 0.01$. Half the solutions are displayed ($d = 0.8$, $\kappa = 3/4$).

The matching process between the inner and outer solutions near $s = 0$ is written as

$$\lim_{\epsilon \rightarrow 0} \left\{ \epsilon \hat{\mathbf{R}}_0 \left(\frac{\eta\tau}{\epsilon} \right) + \epsilon^2 \hat{\mathbf{R}}_1 \left(\frac{\eta\tau}{\epsilon} \right) + \dots \right\} = \lim_{\epsilon \rightarrow 0} \{ \mathbf{R}_0(\eta\tau) + \epsilon \mathbf{R}_1(\eta\tau) + \dots \} \quad (34)$$

and leads to the $O(1)$ and $O(\epsilon)$ conditions:

$$0 = \mathbf{R}_0(0), \quad (35a)$$

$$0 = \mathbf{R}_1(0). \quad (35b)$$

A similar matching process between the reflected inner and outer solutions near $s = 1$ yields

$$d\mathbf{i} = \mathbf{R}_0(1), \quad (36a)$$

$$0 = \mathbf{R}_1(1). \quad (36b)$$

Eq. (35a) determines (X_0, Y_0, Z_0) , whereas Eq. (36a) provides a set of three nonlinear equations for the determination of \mathbf{C}_0 . There are two solutions: a tensile solution with $c_x < 0$ and a compressive solution with $c_x > 0$. As typically the $c_x < 0$ solution is the physically realised one, only this solution is considered in the present study. Eqs. (35a) and (36a) then provide six linear equations for the components of \mathbf{C}_1 and the integration constants (X_1, Y_1, Z_1) , which are all found to be zero. We therefore have that the $O(\epsilon)$ outer solution $\mathbf{R}_1 \equiv \mathbf{0}$. This information is used in Eq. (31) to obtain the $O(\epsilon)$ and $O(\epsilon^2)$ terms of the boundary layer solution.

Fig. 2 shows a comparison of the analytical results with ‘exact’ results obtained by numerically integrating Eq. (9). The solid line shows the results of the numerical calculations, whereas the long dashed line shows the two-term outer solution, and the dotted line shows the two-term inner solution. In order to bring out the boundary layer more clearly, the derivatives of the various solutions rather than the solutions themselves are shown. The agreement is found to be reasonable for $\epsilon = 0.05$ (Fig. 2a) and would be better for smaller values of ϵ such as $\epsilon = 0.01$ (Fig. 2b).

4. High-twist analysis

In order to solve Eq. (9), it is useful to introduce an angular description of the tangent vector to the rod, which automatically satisfies the inextensibility constraint. A convenient choice in order to avoid polar

singularities for the present configuration is to choose the Euler angles θ and ϕ , which are defined in terms of Cartesian coordinates (x, y, z) via

$$\begin{aligned}\mathbf{R} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \\ \mathbf{R}' &= \cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} + \sin \theta \mathbf{k}.\end{aligned}\quad (37)$$

The variable ϕ represents the angle the tangent makes with the horizontal plane, whereas $\sin \theta$ measures the tangential component in the z direction.

The expansion of the vector \mathbf{C} in terms of Cartesian components,

$$\mathbf{C} = C_x \mathbf{i} + C_y \mathbf{j} + C_z \mathbf{k}$$

allows Eq. (9) to be expressed as a system of seven first-order differential equations for $(x, y, z, \phi, \theta, \phi', \theta')$. These are given by

$$\begin{aligned}\phi'' &= 2\phi' \theta' \tan \theta - \frac{2\pi\kappa Tw \theta'}{\cos \theta} - \frac{C_x \sin \phi - (C_y - s) \cos \phi}{\varepsilon^2 \cos \theta}, \\ \theta'' &= \theta'^2 \tan \theta + \left[\frac{F(s)}{\varepsilon^2} - G(s) \right] \tan \theta + 2\pi\kappa Tw \phi' \cos \theta + \frac{C_z}{\varepsilon^2 \cos \theta},\end{aligned}\quad (38)$$

where the functions $F(s)$ and $G(s)$ are defined by

$$F(s) = s \cos \theta \sin \phi - (C_x \cos \theta \cos \phi + C_y \cos \theta \sin \phi + C_z \sin \theta) G(s) = \theta'^2 + \phi'^2 \cos^2 \theta, \quad (39)$$

together with the definitions of Eq. (37):

$$x' = \cos \theta \cos \phi, \quad y' = \cos \theta \sin \phi, \quad z' = \sin \theta. \quad (40)$$

The boundary conditions (10) can be expressed in terms of the seven dependent variables in the obvious way, leading to ten conditions, five at the left end and five at the right. Given that there are also three unknown constants (C_x, C_y, C_z) , this provides a formally well-posed boundary-value problem that can be solved, for example, using a shooting method.

A further reduction is possible due to the assumed symmetry of the shape about $s = 1/2$. The Eqs. (38)–(40) are invariant under the reversing symmetry

$$\begin{aligned}x(s) &= d - x(1 - s), & y(s) &= y(1 - s), & z(s) &= -z(1 - s), \\ \phi(s) &= -\phi(1 - s), & \theta(s) &= \theta(1 - s), & s &\in [0, 1],\end{aligned}\quad (41)$$

and from Eq. (38) this implies

$$C_y = \frac{1}{2}. \quad (42)$$

For the rod shapes which obey Eq. (41), one can pose a simpler boundary-value problem with s running over $[0, 1/2]$ by using

$$x(0) = y(0) = z(0) = 0, \quad x(1/2) = d/2, \quad z(1/2) = 0, \quad \phi(1/2) = 0, \quad \theta'(1/2) = 0. \quad (43)$$

The use of Eq. (43) in place of Eq. (10) results in one less boundary condition and one less scalar unknown as C_y is now fixed by the symmetry. Hence, the problem remains formally well posed, but now has the computational advantages of having to be solved on a smaller domain and being satisfied by a smaller class of solutions within a potential multiplicity.

4.1. Numerical results

The system of Eqs. (38) and (40), subject to the boundary conditions (43), has been solved using the software package AUTO (Doedel et al., 1997)¹ for the specified values of d , ϵ , κ , and Tw . This software uses a collocation method, which automatically refines grids to capture regions, where the solution changes rapidly, such as the boundary layers at the ends of the rod.

The purpose of this section is to study what happens to the catenary-like solution as Tw is increased. All results are presented for a fixed value $\epsilon = 0.01$. Fig. 3 shows a series of profiles computed under increasing Tw for the span lengths $d = 0.99, 0.95$, and 0.80 . In each case, the shape remains essentially in the low-twist configuration until a critical twist is reached, at which point, the sag begins to decrease as a superimposed helix develops onto the sagged span. As Tw continues to increase, the number of helical turns in the outer region increases and to preserve total arclength, the span centre rises up and the helical radius becomes smaller. Fig. 4 shows plots of the associated end tension $T(0)$ and the magnitude of the centre sag $|y(1/2)|$ versus the applied twist Tw for the three values $d = 0.99, 0.95$, and 0.80 .

From Figs. 3 and 4, it is clear that the low-twist configuration remains essentially unperturbed until a critical value of Tw is reached. As there is a gravitational force, the tension in the rod is not constant, so that the Greenhill buckling analysis does not apply. However, the nondimensional form of the Greenhill buckling formula for a finite-length rod has the form

$$4\pi^2\epsilon^4 = Q^2 - 4\epsilon^2 T(0),$$

where the end tension $T(0)$ has been used. The torque Q is eliminated in terms of the twist Tw with Eq. (9), and the resulting estimate for the critical twist Tw_c for the Greenhill-like buckling is

$$\text{Tw}_c = \sqrt{\frac{1}{\kappa^2} + \frac{T(0)}{\kappa^2\pi^2\epsilon^2}} \sim \frac{\sqrt{T(0)}}{\kappa\pi\epsilon}.$$

The estimates of the critical twist Tw_c for the span lengths $d = 0.99, 0.95$, and 0.80 based on the tension $T(0)$ associated with the zero-twist solution (i.e. $\text{Tw} = 0$) are, respectively, 61, 42 and 33. This comparison to the values 61, 42, and 35 at the sharp transition points in Fig. 4. At the transition points in Fig. 4, the nature of the solution in the outer region begins to change. This can be seen in the associated rod shapes of Fig. 3. The post-transition solution consists of a catenary-like solution across the span with a superimposed helix-like solution which has a period of $O(\epsilon)$.

As Tw continues to increase, the number of helical turns increases, so that the sag and helical amplitude both decrease to preserve total arclength.

Notice that the helix-like solution superimposed on the catenary in general appears to be quasiperiodic. Moreover, after an initial jump at the transition point, the amplitude of the helix decreases with Tw . These effects can be seen more clearly in Fig. 5, which shows in details a computation for $d = 0.98$. In fact, after the initial jump, the helix amplitude (which can be estimated by taking the average of the oscillations in Fig. 5b) appears to behave roughly as the square root of the sag $|y(d/2)|$ depicted in Fig. 5a. Also, note that the oscillations in the maximum of $z(s)$ indicate that the solution is a modulated helix with the maximum amplitude of the modulation rotating in the y and z planes as Tw is increased. This is further emphasised by Fig. 5c and e, which shows solutions that are approximately quasiperiodic. However, at certain isolated values of Tw , the frequency of the helix and its modulation may hit resonance resulting a solution that is approximately periodic as indicated in Fig. 5d and f.

¹ Available by anonymous ftp from ftp.cs.concordia.ca, directory pub/doedel/auto.

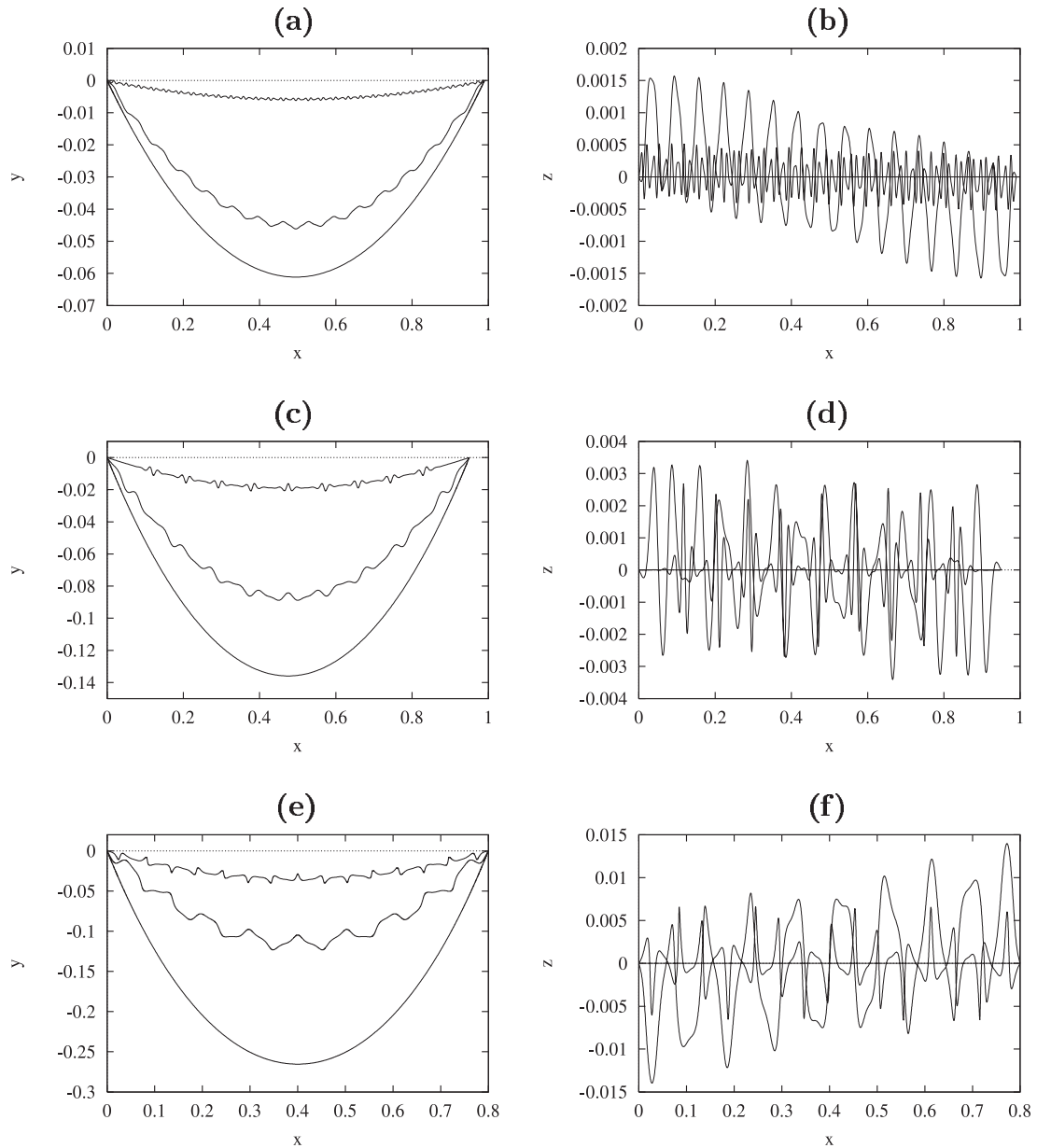


Fig. 3. xy and xz projections of the hanging rod at $d = 0.99$ (top), 0.95 (middle) and 0.80 (bottom) ($\epsilon = 0.01, \kappa = 3/4$). In (a) and (b), $Tw = 0, 80, 100$; in (c) and (d), $Tw = 0, 50, 100$; in (e) and (f), $Tw = 0, 40, 60$.

For lower values of d (e.g. $d = 0.95$ and 0.8 illustrated in Fig. 3b and c), it is apparent that as Tw is further increased, the helix-like response develops a localised profile. That is, it develops the characteristics of a chain of wavepackets. For Tw values higher than those depicted, it became numerically problematical to continue the branch of solutions, but there was an evidence to suggest that the localization process continues until presumably the rod starts to self-intersect and writhe into a double helix. Some of the causes of these numerical difficulties shall be commented in Section 5.

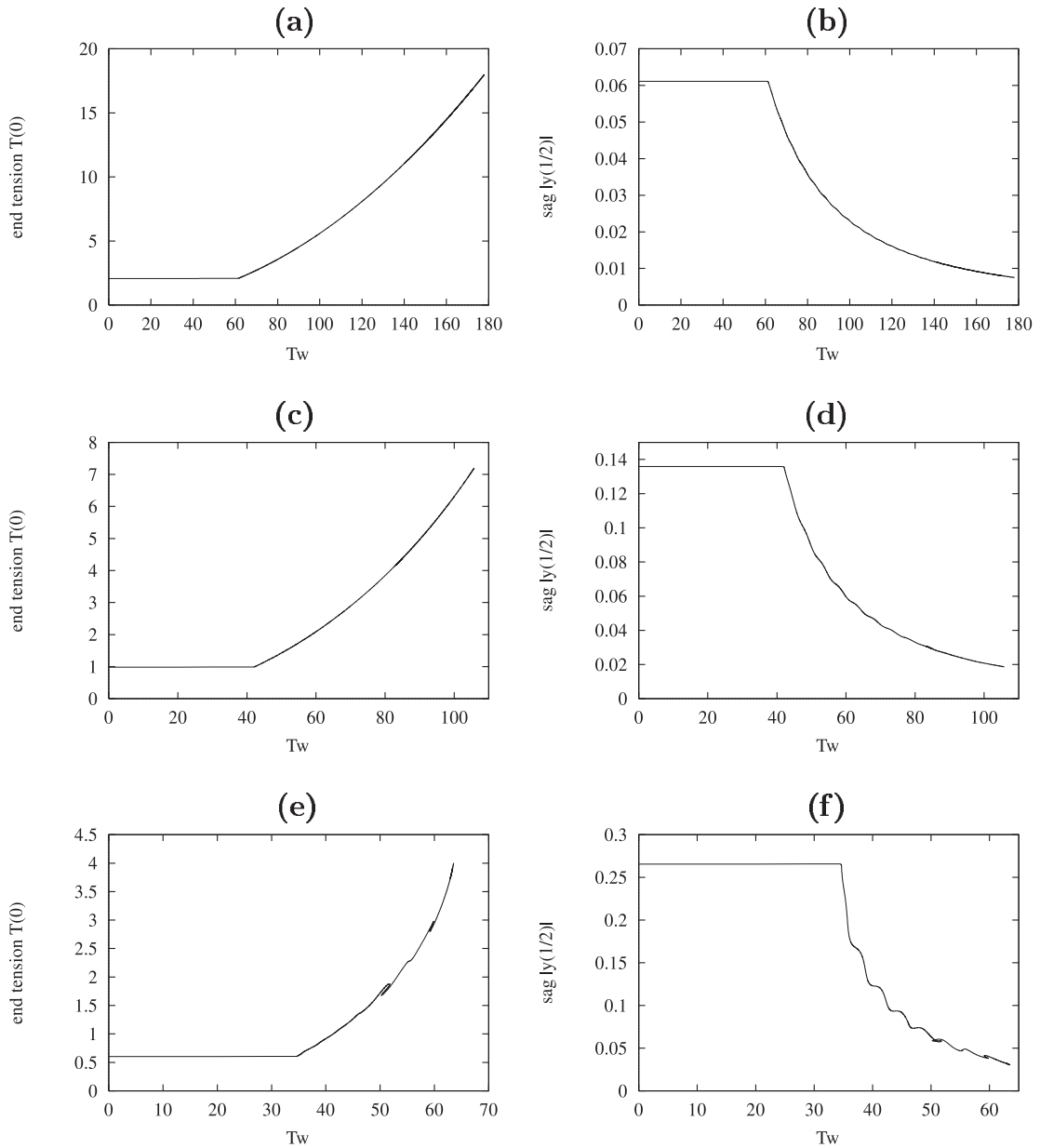


Fig. 4. End tension $T(0)$ and sag $|y(1/2)|$ against total twist Tw at $d = 0.99$ (top), 0.95 (middle) and 0.80 (bottom) ($\epsilon = 0.01$, $\kappa = 3/4$).

Instead, Fig. 6 presents the results for the less severe value of $\epsilon = 0.1$, where it is possible to compute to the limit of large Tw more reliably. Note from Figs. 6c and d, the behaviour for $d = 0.98$. Compared with the results for $\epsilon = 0.01$, there is less distinction between the low-twist and high-twist regimes and the sharp transition at $Tw = Tw_c$ has been smoothed out, as should be expected because ϵ is now of moderate size. As Tw is increased, only a small number of helical waves are fitted in, before a marked localisation takes place towards the centre of the rod. At $Tw = Tw_j \approx 40.7$, the localisation is so extreme that the sag of the ca-

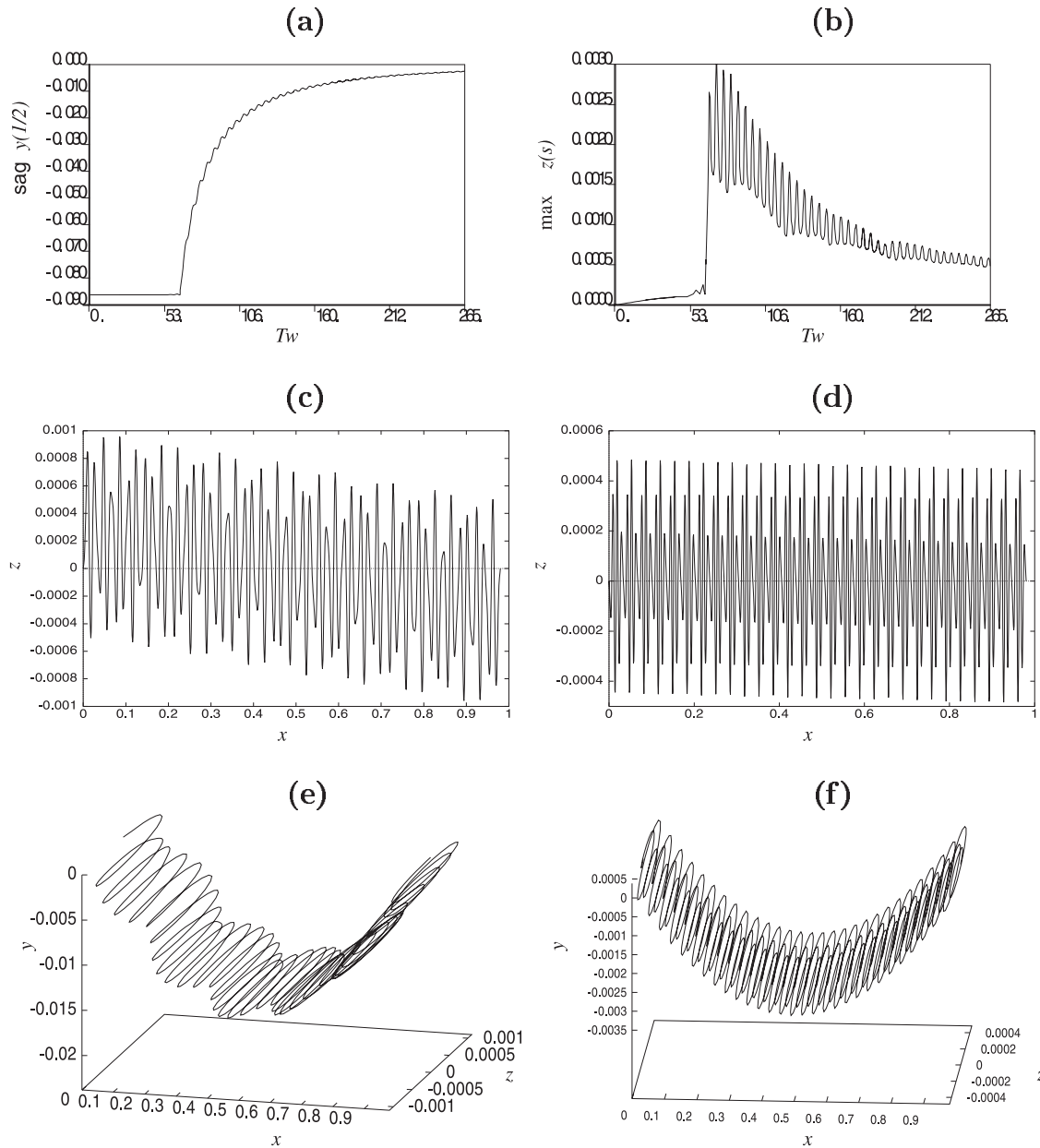


Fig. 5. Further details of a computation for $d = 0.98$ ($\varepsilon = 0.01$, $\kappa = 3/4$). (c) and (d) show projections of solutions at $Tw = 170$ and 265 respectively and (e) and (f) show 3D rod shapes at $Tw = 212$ and 265 , respectively.

tenary has almost disappeared and, precisely at the mid-point of the rod, the tangent vector lies in the z direction. It is problematical to compute the branch for any higher Tw values, because the polar representation (37) now breaks down. However, given that the loading is approximate in the x and y planes, then the condition of Coyne (1990) states that at such a point, where the tangent points directly out of the plane, the rod would dynamically jump into a self-intersecting solution. Upon further increase of Tw writhing into

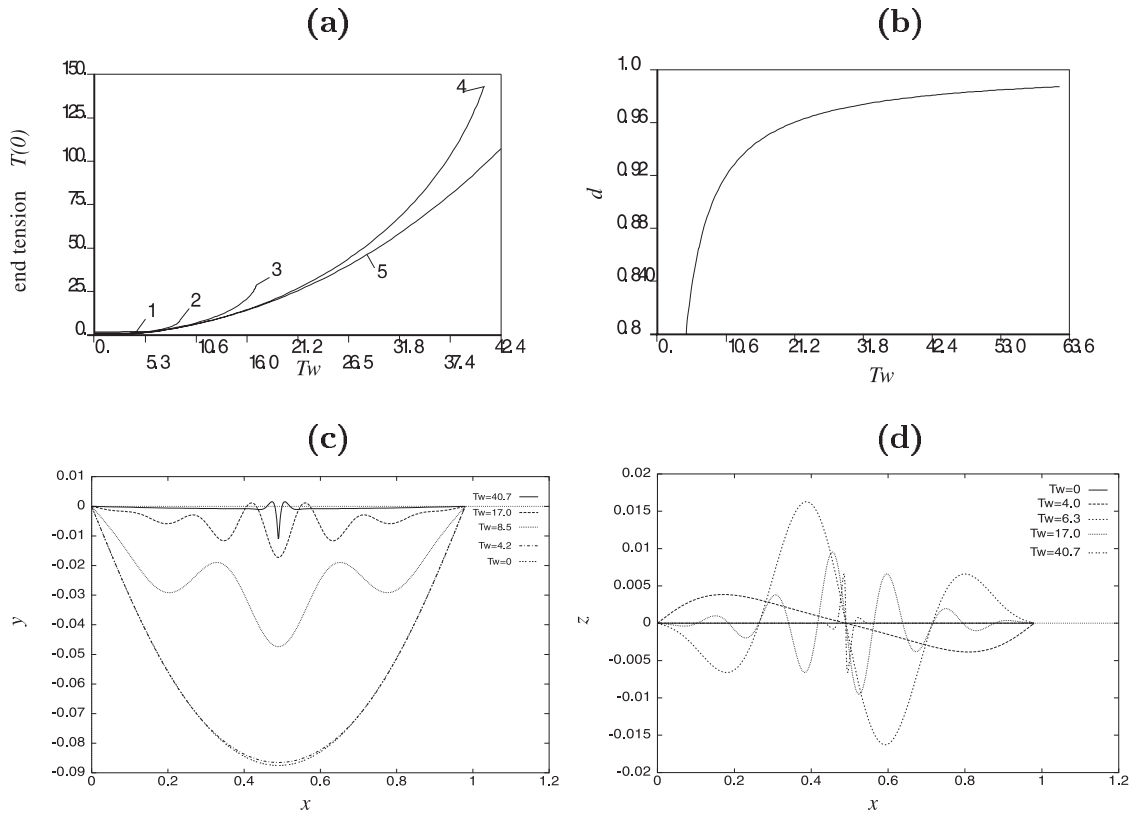


Fig. 6. Results for $\varepsilon = 0.1$ ($\kappa = 3/4$). (a) End tension $T(0)$ against total twist Tw for (1) $d = 0.8$; (2) $d = 0.9$; (3) $d = 0.95$; (4) $d = 0.98$; (5) $d = 0.99$. Branches (1)–(4) end at ‘jump points’ where the rod points entirely in the z direction; branch (5) can be continued beyond the scale depicted. (b) Locus of jump points in the (Tw, d) plane. (c), (d) xy and xz projections along the curve for $d = 0.98$ at the annotated Tw values.

a double helical ply plus an end loop (as described in Stump et al., 1998) would presumably occur. Fig. 6(b) shows how Tw_j varies with d for $\varepsilon = 0.1$. Note that a degree of sensitivity to d was found in determining whether the localisation ‘writhes down’ (with $y(1/2) < 0$) as in Fig. 6(c) or ‘writhes up’ (with $y(1/2) > 0$).

4.2. Multi-scaling analysis

The numerical results for the shallow hanging rod show that a sudden transition in behaviour occurs for Tw of $O(1/\varepsilon)$. The governing equation (9) continues to apply, but the separation of length scales between the inner and outer layers which occurs for $Tw \sim O(1)$ does not hold for $Tw \sim O(1/\varepsilon)$. Instead, the numerical results suggest that the outer solution develops a multiple-scale character consisting of a catenary-like portion on the scale s , and a superimposed helical solution with a wavelength of $O(\varepsilon)$. This suggests a multi-scaling analysis in which $\mathbf{R}'(s)$ is replaced with $\mathbf{F}(s, \xi)$ and derivatives with respect to s are replaced by

$$\frac{d}{ds} = \frac{\partial}{\partial s} + \frac{1}{\varepsilon} \frac{\partial}{\partial \xi}.$$

With the definition of the $O(1)$ variable

$$q = 2\pi\kappa\epsilon Tw,$$

the governing equation (9) becomes

$$q\epsilon\mathbf{F} \times \mathbf{F}_s + q\mathbf{F} \times \mathbf{F}_\xi - \epsilon^2\mathbf{F}_{ss} - 2\epsilon\mathbf{F}_{s\xi} - \mathbf{F}_{\xi\xi} - \epsilon^2\mathbf{F}(\mathbf{F}_s \cdot \mathbf{F}_s) - 2\epsilon\mathbf{F}(\mathbf{F}_s \cdot \mathbf{F}_\xi) - \mathbf{F}(\mathbf{F}_\xi \cdot \mathbf{F}_\xi) + \mathbf{F}[s(\mathbf{F} \cdot \mathbf{j}) - \mathbf{F} \cdot \mathbf{C}] - s\mathbf{j} + \mathbf{C} = \mathbf{0}, \quad (44)$$

where the subscripts denote differentiation with respect to the variable. This equation is sufficiently complicated that we confine our interest to the nature of the outer solution to the leading order and attempt only to separate the leading-order behaviour on the different scales.

Let us formally suppose that

$$\mathbf{F} = \mathbf{H}(s) + \mathbf{G}(\xi), \quad (45)$$

where for the sake of book-keeping, both \mathbf{H} and \mathbf{G} are assumed to be $O(1)$ (although we shall see later that this assumption is not necessary). Then, leading order (44) becomes

$$O(1) : q(\mathbf{H} + \mathbf{G}) \times \dot{\mathbf{G}} - \ddot{\mathbf{G}} - (\mathbf{H} + \mathbf{G})(\dot{\mathbf{G}} \cdot \dot{\mathbf{G}}) + (\mathbf{H} + \mathbf{G})[s(\mathbf{H} + \mathbf{G}) \cdot \mathbf{j} - (\mathbf{H} + \mathbf{G}) \cdot \mathbf{C}] - s\mathbf{j} + \mathbf{C} = 0, \quad (46)$$

where an over dot denotes differentiation with respect to ξ . Now, if we treat s and ξ as independent coordinates, Eq. (46) must be interpreted differently on the two different length scales. On the *long* scale \mathbf{G} and its derivatives may be replaced by their averages over the shorter length scale. Based on the numerical evidence, we seek solutions for \mathbf{G} that are periodic or quasiperiodic perturbations about some fixed vector. Hence, we consider

$$\langle \mathbf{G} \rangle = \mathbf{D}, \quad \langle \dot{\mathbf{G}} \rangle = \mathbf{0}, \quad \langle \ddot{\mathbf{G}} \rangle = 0,$$

where $\langle \cdot \rangle$ represents the average value, and \mathbf{D} is an unknown constant vector. Upon setting $\tilde{\mathbf{H}} = \mathbf{H} + \mathbf{D}$, Eq. (46) then becomes

$$\tilde{\mathbf{H}}[s(\tilde{\mathbf{H}} \cdot \mathbf{j}) - \tilde{\mathbf{H}} \cdot \mathbf{C}] - s\mathbf{j} + \mathbf{C} = 0. \quad (47)$$

This equation up to boundary conditions and constant of integration \mathbf{C} is precisely equivalent to the $O(1)$ part of Eq. (12a) and (12b) whose solution (13) describes the outer catenary of the untwisted rod. As we are interested in almost flat solutions, for which the span d is close to one, we expand the constant,

$$\mathbf{C} = -\tilde{T}\mathbf{i} + C_y\mathbf{j} + \text{h.o.t.},$$

where C_y is $O(1)$, and \tilde{T} is the as yet unknown constant rod tension.

In contrast, on the *short* length scale, we consider some specific arclengths s_0 ($0 < s_0 < 1$) along the rod and replace s by $s_0 + \epsilon\xi$. Then, leading order (46) becomes

$$O(1) : q\tilde{\mathbf{G}} \times \dot{\tilde{\mathbf{G}}} - \ddot{\tilde{\mathbf{G}}} - \tilde{\mathbf{G}}(\dot{\tilde{\mathbf{G}}} \cdot \dot{\tilde{\mathbf{G}}}) - \tilde{\mathbf{G}}[\tilde{\mathbf{G}} \cdot \tilde{\mathbf{C}}] + \tilde{\mathbf{C}} = 0, \quad (48)$$

where $\tilde{\mathbf{C}} = \mathbf{C} - s_0\mathbf{j}$ and $\tilde{\mathbf{G}} = \mathbf{G} + \mathbf{H}(s_0)$ with $\mathbf{H}(s_0)$ regarded as a constant vector. But Eq. (48) is the derivative of the dimensionless equation for a free rod subject to a terminal force $\tilde{\mathbf{C}}$ and twisting moment q , equivalent for example to Eq. 4.1 of Stump et al. (1998). As this equation is posed over the short length scale, the rod is long ($L = 1/\epsilon$). The rod is also naturally straight as Eq. (48) is autonomous in ξ . Now, we assume that the precise effects of the boundary conditions are negligible other than to apply a remote torque q and the unknown remote tension \tilde{T} .

Eq. (48) possesses a rich variety of postbuckling solutions for $q^2 < 4\tilde{T} + O(\epsilon^2)$ (cf. Eq. (1), where $B = 1$ due to nondimensionalisation). For an up to date account of all such solutions, including helices,

modulated helices and localised helices, see van der Heijden and Thompson (1999). The simplest nontrivial solution is the helix which is given by

$$\tilde{\mathbf{G}} = \cos \beta \mathbf{e}_1 + \cos \psi(\xi) \sin \beta \mathbf{e}_2 + \sin \psi(\xi) \sin \beta \mathbf{e}_3,$$

where \mathbf{e}_1 is along the direction of the loading axis and \mathbf{e}_2 and \mathbf{e}_3 are chosen so as to form an orthonormal basis, and the angular description of the helix satisfies

$$\psi = \sqrt{\tilde{T}} \xi, \quad 1 + \cos \beta = q/\sqrt{\tilde{T}}.$$

From these expressions, we can calculate the end-shortening \mathcal{E} associated with the helix

$$\mathcal{E} = (1 - \cos \beta)L = \left(2 - q/\sqrt{\tilde{T}}\right)/\varepsilon, \quad (49)$$

and the amplitude a of the helix given by

$$a = (1/q) \sin \beta(1 + \cos \beta) = \frac{q}{\tilde{T}} \sqrt{2 \frac{\sqrt{\tilde{T}}}{q} - 1}. \quad (50)$$

We now return to the long scale equation (47) and, remembering that we have taken $1 - d$ as small, we write the solution

$$\tilde{\mathbf{H}} = \frac{\tilde{T}}{\sqrt{\tilde{T}^2 + \left(s - \frac{1-\mathcal{E}}{2}\right)^2}} \mathbf{i} + \frac{\left(s - \frac{1-\mathcal{E}}{2}\right)}{\sqrt{\tilde{T}^2 + \left(s - \frac{1-\mathcal{E}}{2}\right)^2}} \mathbf{j} + \text{h.o.t.},$$

where we have evaluated the end condition at $s = 1 - \mathcal{E}$ rather than at 1. The physical shape is obtained by evaluating $\mathbf{R} = \int \mathbf{H}(s)$, which leads to the end condition,

$$d = 2\tilde{T} \sinh^{-1} \left(\frac{1 - \mathcal{E}}{2\tilde{T}} \right), \quad (51)$$

and the sag $|y((1 - \mathcal{E})/2)|$ given by

$$\text{sag} = \sqrt{\tilde{T}^2 + \frac{(1 - \mathcal{E})^2}{4}} - \tilde{T}.$$

Now, Eq. (51), given Eq. (49), is a nonlinear equation for determining \tilde{T} in terms of ε , q and d . In the limit of large \tilde{T} , the solution to this equation can be approximated by

$$\tilde{T} = \frac{q^2}{2} + \text{h.o.t.} = \pi^2 \kappa^2 \varepsilon^2 \text{Tw}^2 + \text{h.o.t.}$$

Back substitution, then gives

$$\mathcal{E} = 1 - d + \text{h.o.t.},$$

$$a = \frac{\sqrt{8\varepsilon(1-d)}}{q} + \text{h.o.t.} = \frac{\sqrt{2(1-d)}}{\sqrt{\varepsilon\pi\kappa}\text{Tw}} + \text{h.o.t.},$$

and

$$\text{sag} = \frac{1}{4} \left(\sqrt{q^4 + 4d^2} - q^2 \right) = \frac{d^2}{8\pi^2 \kappa^2 \varepsilon^2 \text{Tw}^2} + \text{h.o.t.}$$

In all of these expressions, h.o.t. refers to the terms which are smaller as $1 - d \rightarrow 0$, $\varepsilon \rightarrow 0$ and $q \rightarrow \infty$.

These results agree with the qualitative scalings that may be deduced from the numerics. In particular, as Tw is increased well beyond Tw_c , the end tension $T(0) = \tilde{T}$ scales like q^2 i.e. $\tilde{T} \propto \varepsilon^2 Tw^2$, and the sag scales like the square of the amplitude of the helix with both tending to zero as $Tw \rightarrow \infty$.

Finally, let us comment on the original assumption that both $\mathbf{H}(s)$ and $\mathbf{G}(\xi)$ were $O(1)$. As we have now seen, we never actually solve equations for these quantities, but instead deal with the modified quantities $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$, both of which are indeed $O(1)$ and are coupled through the constants \mathbf{D} and $\tilde{\mathbf{C}}$. In fact, there is some indeterminacy in what part of the solution for \mathbf{F} in the \mathbf{i} direction is regarded as belonging to \mathbf{H} and what part belonging to \mathbf{G} .

5. Discussion

In this paper, we have demonstrated, via close agreement between numerics and asymptotics, two distinct regimes of a rod hanging under gravity subject to an applied twist. For the rods of fixed material properties, the dimensionless parameter ε may be regarded as the aspect ratio of the rod, with $\varepsilon \rightarrow 0$ being the limit of a long thin rod. For the twist of order much less than this aspect ratio, the effect of applying torsion to a hanging catenary is negligible except in boundary layers of proportion $O(1/\varepsilon)$ of the rod's length. The present study has considered simply supported ends, in which case the boundary layer leads to sharp transitions only in the rate of change of the rod curvature. This contrasts to the situation for clamped boundary conditions, where the analysis of Stump and van der Heijden (2000) (which albeit had a different overall geometry) shows that the sharp transitions occur in the rod tangent.

We have then seen that at approximately the classical Greenhill buckling twist (suitably rescaled) $Tw = Tw_c \approx \sqrt{\tilde{T}}/(\pi\kappa\varepsilon)$, the response changes markedly over a short range of Tw to a catenary with a super-imposed helical deformation. The process of this transition we have not described in detail, but is akin to an imperfect bifurcation, with either ε or $1-d$ playing the role of the imperfection parameter.

The second regime holds for $Tw > Tw_c$, and we have been able to motivate what was observed in careful numerical experiments via a heuristic double-scale analysis. This reveals that by looking on the appropriate length scales, one recovers either the equation for a free rod subject to tension and torsion or that for a torsion free rod hanging under gravity. The former is known to be completely integrable and admit solutions in the form of helices and modulated helices. The latter has the well-known catenary solution. These solutions were then matched by assuming that the helix supplies an effective end shortening to the arclength of the catenary. This then enabled us to uncover the numerically suggested scalings for the end tension, the sag of the catenary and the amplitude of the superimposed catenary as $Tw \rightarrow \infty$.

However, the numerics reveal that one cannot pass to the limit $Tw \rightarrow \infty$, and in fact, there is a third regime, where the helical response (which is in general a modulated periodic waves) localises to the extent that itself intersects. Presumably, for higher Tw , the rod will go on to writhe into double helical plies joined by end loops as is known to occur for the gravity-free problem, although such solutions are beyond the scope of the present analysis. The evidence from Fig. 6 suggests that this writhing regime begins for $Tw = O(1/(\varepsilon(1-d)))$.

Another limitation of the double-scale asymptotics is that they do not capture the precise features of the helical solutions on the small scale. These appear in general as quasiperiodic modulated helices rather than a purely periodic response as assumed in the analysis.

Some comments are in order on the accuracy and repeatability of the numerical results. All presented bifurcation curves were computed using AUTO with error tolerances $O(10^{-6})$ and typically 200 collocation intervals with four collocation points in each interval. The differential equations were posed on the half-interval $0 < s < 1/2$ using the symmetry conditions (43). Preliminary numerical experiments over the full interval $0 < s < 1$ revealed qualitatively similar results, but the computations were found to be far less robust. Typically numerical continuation would break down for lower Tw values (even after doubling the

number of collocation intervals in AUTO) and would be characterised by a precursive jump to a symmetry-broken state before nonconvergence was encountered.

Even using the half-interval formulation, problems were sometimes encountered with nonrepeatability of numerical results in resolving the fine features of solutions for $Tw > Tw_c$. Loops in the postbuckled path as in Fig. 4e and f could sometimes be missed unless stepsize tolerances were chosen small enough. Also, as mentioned at the end of Section 4.1, for the five branches showed in Fig. 6a, where $\varepsilon = 0.1$, it was sometimes found that the rod writhed up and sometimes writhed down. It would appear that there are multiple possible solution paths in the high-twist regime, but only one of these is the fundamental path, which is connected to the boundary layer solution as $Tw \rightarrow 0$. For smaller ε , like the solutions showed in Fig. 3e and f, it appears to be many points at which the rod's deflection is localising, yet in a physical experiment self-intersection and a subsequent jump would occur at only one point along the rod's length. Clearly, there is a sensitivity to the boundary conditions, especially to the value of d .

While we have focused only on simply supported boundary conditions, preliminary numerical experiments have been performed in the case of a clamped rod. A similar high-twist region was uncovered, with helices superimposed on an underlying catenary, but the numerical sensitivity seemed even more problematical, perhaps due to the sharper boundary layers.

One message of the present work is that, contrary to the assumptions of Champneys et al. (1997), it is not reasonable to assume that gravity is a small perturbing parameter except in the case of high initial tension $T(0)$ of the untwisted rod (i.e. d close to 1). For example, contrast the findings of the present study with what would happen in the gravity free problem of a long straight rod under dead loading, where applied twist Tw is held constant and the ends shortened while remaining under tension (see Fig. 14 of Thompson and Champneys, 1996). There, the initial postbuckling response would be a localised helix (that is, one for which the modulation has infinite period). As end shortening was further increased, this localisation quickly become more severe with the rod jumping into a writhing configuration sooner. In the present analysis, where admittedly Tw is now the control parameter, before such localisation and writhing can take place, the rod must take up the sag in the catenary, which requires a large amount of extra applied twist. During this process, the buckling response is not a localised helix but a periodically modulated one.

Clearly, the present study is only a first step towards a full understanding of the behaviour of twisted rods hanging under gravity. Nevertheless, it is one that concurs qualitatively with phenomenological experiments (see Fig. 6 of Thompson and Champneys, 1996). Future work should include, a quantitative comparison with experiments, an exploration of the transition as an imperfect bifurcation, including the possibility of multiple solution branches, proper consideration of other boundary conditions such as clamped, and the effects of other 'perturbing' influences like intrinsic curvature, anisotropy or axial extensibility of the rod.

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